

Constrain the Dark Matter Electron Cross Section from Pulsating White Dwarfs

Jia-Shu Niu,^{1,2,*} Weikai Zong,³ and Hui-Fang Xue³

¹Key Laboratory of Theoretical Physics, Institute of Theoretical Physics,
Chinese Academy of Sciences, Beijing, 100190, P.R.China

²School of Physical Sciences, University of Chinese Academy of Sciences,
No.19A Yuquan Road, Beijing 100049, P.R.China

³Astronomy Department, Beijing Normal University, Beijing 100875, P.R.China

(Dated: March 30, 2017)

We propose a novel and feasible method to detect dark matter (DM) electron interaction via pulsating white dwarfs (WDs) in the central region of globular clusters. Annihilation of the DM particles captured by those WDs can provide additional energy source along the natural cooling evolution of WDs and the cooling print can be well offered by precise asteroseismology. The measurement of the long time scale physical quantity – the rates of period variation of pulsation modes – could be used to constrain the cross section between DM particles and electrons ($\sigma_{\chi,e}$), when DM particle mass $m_\chi \gtrsim 5$ GeV. We construct estimations to prove that this original method is feasible and can be implemented in the challenging time-series photometry in the near future.

Introduction.—Although dark matter (DM) contributes to 26.8% of the total energy density of the Universe [1], the particle nature of DM remains largely unknown. The most popular candidates are weakly interacting massive particles (WIMPs) for which are currently searched with different strategies, i.e., direct detection, indirect detection and detection on large terrestrial colliders (see, e.g., [2] for reviews). The most stringent constraints from experiments of cross section are currently, for proton, $\sigma_{\chi,p} \lesssim 1.1 \times 10^{-46} \text{ cm}^2$ at a WIMP mass of $m_\chi = 50 \text{ GeV}/c^2$, and for electron, $\sigma_{\chi,e} < 3 \times 10^{-38} \text{ cm}^2$ at $m_\chi = 100 \text{ MeV}$ and $\sigma_{\chi,e} < 10^{-37} \text{ cm}^2$ when $20 \text{ MeV} \leq m_\chi \leq 1 \text{ GeV}$ [3, 4].

An alternative search for DM probe can be upon on celestial objects, such as stars, that have huge volume and large mass, comparing to the manual facilities. There are some branches collaborators who had began working on: the Sun and main-sequence stars, considered the precise properties of their interior structures from helioseismology and asteroseismology (see, e.g., [5–7]); and the compact stars (i.e., white dwarfs and neutron stars), considered their deficiency of nuclear energy source (see, e.g., [8–11]). Nevertheless, all these attempts are attracted on the interactions between WIMPs and nucleons whose cross section ($\sigma_{\chi,p}$) has been constrained below $1.1 \times 10^{-46} \text{ cm}^2$ by direct detection [3]. In order to enhance the interacting effects between WIMPs and nucleus over the entire stars, a circumstance with high DM local density is preferred, where the stars in such as galaxy center, globular cluster and dwarf galaxy can be a reasonable solution.

In this letter, we consider the hydrogen-atmosphere pulsating white dwarf stars (WDs) in the central region of globular clusters such as ω Cen, and focus on the DM-electron interactions. The hydrogen-atmosphere (DA) WDs consist of $\sim 80\%$ WDs sample that are the ending fates of $\sim 98\%$ stars in the Galaxy. Precise asteroseismology on DA variables (DAVs) can reveal their interior

structures and determine the secular variations of period variation of oscillation modes spanning over long time scales (see, e.g., [12, 13] for reviews). The secular rates can be used to determine the evolutionary cooling rate of WDs. However, the DM captured by WDs in a local dense DM environment possibly affects the cooling rates of WDs predicted by standard cooling model (SCM). In reverse, we can use the observed WD cooling rates to constrain the cross section in a condition with known DM density. Moreover, WDs are the most electron-dense objects and can be the best laboratory to measure the DM-electron cross section ($\sigma_{\chi,e}$).

The capture rate of WIMPs in WDs.—Galactic WIMPs are inevitably streaming through any celestial object. Those particles will loose energy when they scatter with nucleons (which we did not mainly considered in this letter) or electrons inside the celestial object, leading their speed down. If the velocity of the WIMP reaches below the escape velocity, the WIMP will be “captured”, i.e. it becomes bound to the star. Regardless of the effect of evaporation which is not important in this letter where we consider the WIMPs mass $m_\chi > 5 \text{ GeV}$ [14, 15], the evolution of the total number of WIMPs, N_χ , inside the star (or any celestial object) can be written as,

$$\dot{N}_\chi = \Gamma_c - 2\Gamma_a, \quad (1)$$

where Γ_c is the particle capture rate, $\Gamma_a = \frac{1}{2}C_a N_\chi^2$ is the annihilation rate. Therefore, we have $N_\chi = \Gamma_c \tau \tanh(\frac{t}{\tau})$ with the equilibrium timescale $\tau = \sqrt{\frac{1}{C_a \Gamma_c}}$. When the dynamic equilibrium state reached, the WIMPs capture rate is balanced by the annihilation one [16], i.e., $\Gamma_c = 2\Gamma_a$.

Considering the limits that almost all WIMPs crossing the star to be captured, the WIMP capture rate is determined by the geometrical limit πR_\star^2 rather than the total interaction cross section $\sigma_{\chi,e} N_e$. We thus use a effective cross section σ_{eff} defined as

$$\sigma_{\text{eff}} = \min \left[\sigma_{\chi,e}, \frac{\pi R_*^2}{N_e} \right], \quad (2)$$

where N_e is the total number of electrons in the WD.

The capture rate for a Maxwell-Boltzmann WIMP velocity distribution (in the observer's frame) by a star moving with an arbitrary velocity v_* relative to the observer is given by Gould [17], but we did some additional simplifications as well: (i) a uniform distribution of matters in a WD: $\rho_*(r) = \rho_* = M_*/V_*$ (ρ_* , M_* and V_* are the density, mass and volume of a WD); (ii) the same chemical composition over the entire scattering volume V_* ; (iii) a uniform temperature profile (calculated from Eq. 10) in WD because of the extremely high thermal conductivity of a electron degenerate core; (iv) as WDs are always electrically neutral we use the value $\frac{1}{2} \frac{M_*}{m_p}$ and $\frac{1}{2} \frac{\rho_*}{m_p}$ as the total number of electrons (N_e) and local number density of electrons (n_e) in a WD, respectively (m_p and m_e are the mass of proton and electron). With these preparations, we can formulate the capture rate in a WD as

$$\Gamma_c = \int_0^{R_*} dr \, 4\pi r^2 \frac{d\Gamma_c(r)}{dV}, \quad (3)$$

where

$$\begin{aligned} \frac{d\Gamma_c(r)}{dV} &= \left(\frac{6}{\pi} \right)^{1/2} \sigma_{\text{eff}} \frac{\rho_*}{2m_p} \frac{\rho_\chi}{m_\chi} \frac{v^2(r)}{\bar{v}^2} \frac{\bar{v}}{2\eta A^2} \\ &\times \left\{ \left(A_+ A_- - \frac{1}{2} \right) [\chi(-\eta, \eta) - \chi(A_-, A_+)] \right. \\ &\left. + \frac{1}{2} A_+ e^{-A_-^2} - \frac{1}{2} A_- e^{-A_+^2} - \frac{1}{2} \eta e^{-\eta^2} \right\} \end{aligned} \quad (4)$$

$$A^2 = \frac{3v^2(r)\mu}{2\bar{v}^2\mu_-^2}, \quad A_\pm = A \pm \eta, \quad \eta^2 = \frac{v_*^2}{2\bar{v}^2}$$

$$\chi(a, b) = \frac{\sqrt{\pi}}{2} [\text{Erf}(b) - \text{Erf}(a)] = \int_a^b dy e^{-y^2}$$

$$\mu_- = (\mu - 1)/2, \quad \mu = m_\chi/m_e$$

, where m_χ and ρ_χ are respectively the WIMP mass and the WIMPs density at the star position. \bar{v} is the WIMPs' speed dispersion of Maxwell-Boltzmann velocity distribution; the velocity of the star moving through the DM halo or sub-halo, labeled as v_* .

The escape velocity at a given radius r inside a star is given by

$$v(r) = \left[2G \int_r^\infty dr' \frac{M(r')}{r'^2} \right]^{1/2} = \left[\frac{GM_*}{R_*} \left(3 - \frac{r^2}{R_*^2} \right) \right]^{1/2}, \quad (5)$$

where $M(r')$ denotes the total mass included in r' .

The Knudsen number, K , indicates the "localization" of the WIMPs transport:

$$K = \frac{l_\chi(0)}{r_\chi}, \quad (6)$$

with $l_\chi(0) = [\sigma_{\chi,e} \cdot n_e(0)]^{-1} = [\sigma_{\chi,e} \cdot n_e]^{-1}$ is the mean free length in the center of the star and $r_\chi = \sqrt{\frac{3kT_c}{2\pi G \rho_c m_\chi}}$ is the typical scale of the DM core in the star (T_c and ρ_c are the temperature and density of the star's core).

Following Griest and Seckel [16], Scott *et al.* [18], Taoso *et al.* [19], in the case of large K (for WDs), the WIMPs' distribution in star can be described by

$$n_\chi(r) = n_\chi(0) \cdot \exp \left[-\left(\frac{r}{r_\chi} \right)^2 \right]. \quad (7)$$

The annihilation term can be computed by a separated way as follows:

$$\Gamma_a = \int_0^{R_*} dr \, 4\pi r^2 \cdot \frac{1}{2} \langle \sigma v \rangle n_\chi^2(r). \quad (8)$$

The factor $1/2$ ($1/4$) in the equation above is appropriate for self (not self) conjugate particles and $\langle \sigma v \rangle$ the velocity-averaged annihilation cross-section.

If an equilibrium between capture and annihilation is reached the annihilation rate reduces to $\Gamma_a = 1/2\Gamma_c$ and it is independent on the annihilation cross-section. With the value of $\langle \sigma v \rangle \simeq 3 \times 10^{-26} \text{ cm}^3 \text{ s}^{-1}$, we can impose $\Gamma_a = 1/2\Gamma_c$ to do the normalization and get $n_\chi(0)$. Thus, the distribution $n_\chi(r)$ is specified and all the related values in this equilibrium state are known.

The rate of period variation of DAVs—The period variation of a DAV is related to two physical processes in the star, the cooling of the star and the contraction of its atmosphere,

$$\frac{\dot{P}}{P} \simeq -a \frac{\dot{T}_c}{T_c} + b \frac{\dot{R}}{R}, \quad (9)$$

where P is the pulsation period for the $m = 0$ multiplet component, T_c is the maximum (normally, core) temperature, R is the stellar radius, and \dot{P} , \dot{T} and \dot{R} are the respective temporal variation rates [12]. The constants a and b are positive numbers of order unity.

From the structure of a WD's envelope, we get:

$$T_0^{\frac{7}{2}} = B \frac{L/L_\odot}{M/M_\odot}, \quad (10)$$

where $B \simeq 1.67 \times 10^{27}$ is a constant and T_0 is the interface temperature between the core and envelope.

If we use the approximation $T_0 \simeq T_c$ (for DAVs) in Eq. 10, substitute the result into Eq. 9 and ignore the mass variation term during the cooling, we can get:

$$\frac{\dot{P}}{P} \simeq -\frac{2a}{7} \frac{\dot{L}}{L}. \quad (11)$$

According to the annihilation of WIMPs in a WD, if the equilibrium state has reached, the luminosity of the WD should be $L_{obs} = L_{mod} + L_\chi$, where L_{obs} is the total observed luminosity of the WD, L_{mod} is the normal luminosity in the SCM and L_χ is the luminosity purely due to the accretion and subsequent annihilation of WIMPs.

From previous section, we get $L_\chi = 2 \cdot \Gamma_a m_\chi c^2 = \Gamma_c m_\chi c^2$. One should note that once the equilibrium state reached, L_χ should not change with time. As a result, $\dot{L}_{obs} = \dot{L}_{mod}$. Thus, the relationships between the rate of period variation and the luminosity should be

$$\frac{\dot{P}_{obs}}{P_{mod}} = \frac{L_{mod}}{L_{mod} + L_\chi}, \quad (12)$$

in which we imposed that the period from model calculation P_{mod} equals the value from observation P_{obs} .

The evolutionary rate of cooling of a WD depends on the stellar mass and core composition, and can be expressed as a function of the mean atomic weight A [20–22]

$$\dot{P}_{mod} = dP/dt = (3 - 4) \times 10^{-15} \frac{A}{14} \text{ s s}^{-1}. \quad (13)$$

In this letter, we use a mean atomic weight 14, which is consistent with the cooling rate of DAVs with a carbon-oxygen core [23].

Estimation.—We perform our estimation in the case that a DAV in the central region of globular cluster. Considering the DAVs G117-B15A as an example to do estimation, the details of its structure are determined by asteroseismology from Romero *et al.* [24], Córscico *et al.* [25], and we list in Tab. I.

Quantity	Asteroseismological model
M_*/M_\odot	0.593 ± 0.007
$\log(R_*/R_\odot)$	-1.882 ± 0.029
$\log(L_*/L_\odot)$	-2.497 ± 0.030
M_{He}/M_*	2.39×10^{-2}
M_H/M_*	$(1.25 \pm 0.7) \times 10^{-6}$
X_C, X_O (center)	$0.28^{+0.22}_{-0.09}, 0.70^{+0.09}_{-0.22}$

TABLE I. M_\odot , R_\odot and L_\odot are the solar mass, radius and luminosity; M_H , M_{He} are the mass of element hydrogen and helium in the star; X_C and X_O are the mass fractions of element carbon and oxygen in the star.

Using Eq. 2 we got $\sigma_{eff} \simeq 7.5 \times 10^{-39} \text{ cm}^2$, which is more than an order of magnitudes lower than the current

upper limit on $\sigma_{\chi,e}$ (Here we assume the 10^{-37} cm^2 upper limit is also available for $m_\chi \gtrsim 5 \text{ GeV}$). This value implies that almost all WIMPs crossing the star are captured by WD as long as we choose $\sigma_{\chi,e} \gtrsim \sigma_{eff}$.

Imposing $r_\chi < R_*$, we get another lower limit of WIMPs' mass ($m_\chi \gtrsim 0.025 \text{ GeV}$) which we considered in this letter. Combined with the previous one, the mass region which we considered in this letter is $m_\chi \gtrsim 5 \text{ GeV}$.

The WIMPs local density of where the WD locates would influence the luminosity of WIMPs' annihilation obviously. In order to get large WIMPs density, we choose ω Cen (a globular cluster) to be the environment to do the estimation. Amaro-Seoane *et al.* [11] has estimated $\rho_\chi \simeq 4 \times 10^3 \text{ GeV cm}^{-3}$ near the center of ω Cen without an intermediate-mass black hole (IMBH) and $4 \times 10^3 \text{ GeV cm}^{-3} \lesssim \rho_\chi \lesssim 4 \times 10^9 \text{ GeV cm}^{-3}$ with an IMBH. We here choose $\rho_\chi = 4 \times 10^3 \text{ GeV cm}^{-3}$ to do the estimation.

From Merritt *et al.* [26], van de Ven *et al.* [27], the members of this cluster are orbiting the center of mass with a peak velocity dispersion $v_* \simeq 7.9 \text{ km s}^{-1}$. Near a IMBH, where orbital motion around a single mass dominates, the test particle (WIMP or star) velocities are Keplerian, $v_* = \bar{v}$.

With the configurations above, we get the DM luminosity of the WD as $L_\chi \simeq 3.2 \times 10^{30} \text{ erg s}^{-1}$, which is of the same order of $L_{mod} \simeq 1.2 \times 10^{30} \text{ erg s}^{-1}$. As a result, the period variation of WD should be $\dot{P}_{obs} \simeq 0.27 \times \dot{P}_{mod} \simeq (0.81 - 1.08) \times 10^{-15} \frac{A}{14} \text{ s s}^{-1} \simeq (0.81 - 1.08) \times 10^{-15} \text{ s s}^{-1}$, which is obviously smaller than the value from SCM [12].

With $\dot{P}_{mod} = 3.2 \times 10^{-15} \text{ s s}^{-1}$ [25], we can present the constraint of $\sigma_{\chi,e}$ by \dot{P}_{obs} in Fig. 1: (a) if $\dot{P}_{obs} < 0.88 \times 10^{-15} \text{ s s}^{-1}$ (depending on the geometrical limit of the star), it gives a lower limit to $\sigma_{\chi,e}$; (b) if $0.88 \times 10^{-15} \text{ s s}^{-1} \leq \dot{P}_{obs} \leq \dot{P}_{mod}$, it gives an upper limit to $\sigma_{\chi,e}$; (c) if $\dot{P}_{obs} > \dot{P}_{mod}$, there should exist other unknown cooling mechanism beyond the SCM. If we include this new cooling mechanism into the SCM and update the value of \dot{P}_{mod} , the conclusions are all available in (a) and (b). We note that in region (b), the real constraint of $\sigma_{\chi,e}$ depends on the systematics of \dot{P}_{mod} . In this case, if we have more precise and reliable SEM and pulsation model for DAVs, we could get more stringent constraint on $\sigma_{\chi,e}$ theoretically.

Moreover, we got $\tau \simeq 8.8 \text{ yr}$, which is really a short period compared with the time scale of WD's formation process. Thus, we can consider WDs are always in the state of equilibrium. The total mass of WIMPs in the WD is about $2 \times 10^{18} \text{ g} \ll M_*$, then its gravitational effects on the WD's interior structure can be ignored.

We also did a similar estimation for WIMPs which captured by the DAVs G117-B15A, according to the interactions with nucleus. In this case, we got $L_\chi \simeq 1.7 \times 10^{24} \text{ erg s}^{-1}$ (here we choose $\sigma_{\chi,p} = 1.1 \times 10^{-46} \text{ cm}^2$),

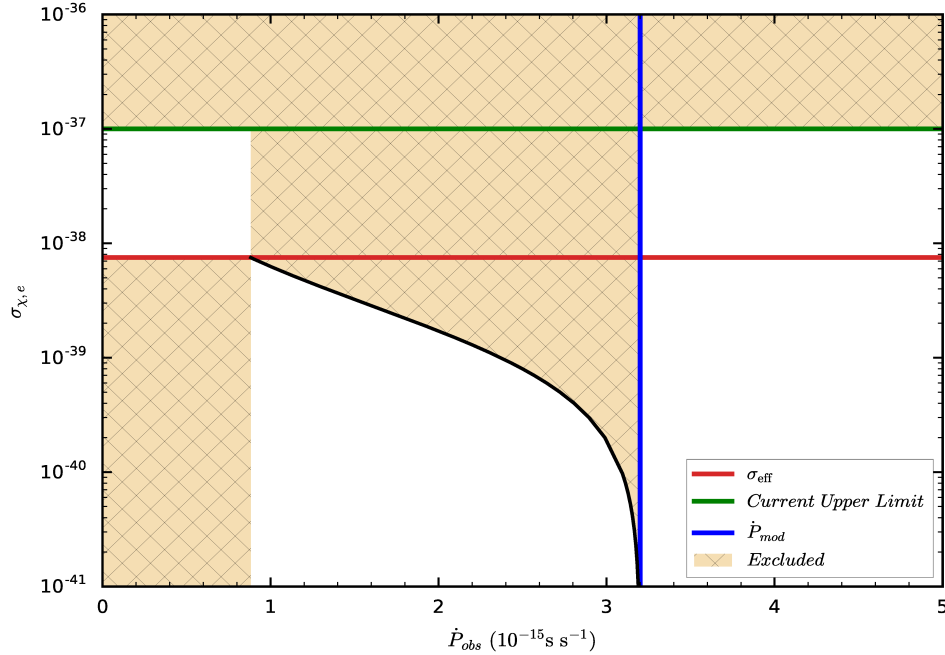


FIG. 1. The excluded region of $\sigma_{\chi,e}$ with observed value of \dot{P}_{obs} . Colored region with slash lines indicate the excluded region with specific \dot{P}_{obs} . Colored lines indicate some important boundaries and threshold. The region $\dot{P}_{obs} > \dot{P}_{mod}$ is not considered in this letter, which we plot it for convenient. Note that \dot{P}_{mod} is the asymptotic line of the black exclusive line. Considering the systematics of \dot{P}_{mod} , the exclusive line always stopped $< \dot{P}_{mod}$ in real cases.

which can be ignored compared with $L_\chi \simeq 3.2 \times 10^{30}$ from electrons. Consequently, it is a reasonable assumption to consider the WIMPs captured by interactions with electrons alone.

Discussion.—In this letter we propose a novel and feasible scenario to measure the DM-electron cross section, a technique independent to the mass of DM particle if $m_\chi \gtrsim 5 \text{ GeV}$. The scenario is based on the assumption that DAV stars are found in a local dense DM environment region such as the central region of globular cluster. According to the measurement of the secular rates of the pulsation periods in those DAVs, we can constrain the DM-electron cross section.

At the same time, if we give a specific value of $\sigma_{\chi,e}$, in this scenario, the observed quantity \dot{P}_{obs} could offer us the WIMPs local density in the position of the DAV. Fig. 2 represents the different profiles of ρ_χ vs. \dot{P}_{obs} with different values of $\sigma_{\chi,e}$. Additionally, ensemble of these DAVs can be used to establish the density profiles in these regions.

From our estimation, we note that once we detect the period variation of such DAVs, it can give us useful information, including: (a) if the rate of period variation is smaller than the threshold (\dot{P}_{gl}) which depend on the geometrical limits of the star (in this our estimation, $\dot{P}_{gl} \simeq 0.88 \times 10^{-15} \text{ s s}^{-1}$), it could provide a lower limit on $\sigma_{\chi,e}$; (b) if $\dot{P}_{gl} < \dot{P}_{obs} < \dot{P}_{mod}$, it could provide a up-

per limit on $\sigma_{\chi,e}$; (c) if $\dot{P}_{obs} \gtrsim \dot{P}_{mod}$, it should exist some other unknown cooling mechanism, like axions (see, e.g., [25, 28–30]).

This scenario can be implemented by high-precision photometry from space mission such as HST, *Kepler* and TESS in the near future.

We thank Tianjun Li and Jun Guo for useful discussion. This research was supported in part by the Natural Science Foundation of China under grant numbers 11135003, 11275246, and 11475238 (TL).

* jsniu@itp.ac.cn

- [1] Planck Collaboration, *Astron. Astrophys.* **571**, A16 (2014), arXiv:1303.5076.
- [2] G. Bertone, D. Hooper, and J. Silk, *Phys. Rept.* **405**, 279 (2005), hep-ph/0404175.
- [3] LUX Collaboration, *Physical Review Letters* **118**, 021303 (2017), arXiv:1608.07648.
- [4] R. Essig, A. Manalaysay, J. Mardon, P. Sorensen, and T. Volansky, *Physical Review Letters* **109**, 021301 (2012), arXiv:1206.2644.
- [5] M. T. Frandsen and S. Sarkar, *Physical Review Letters* **105**, 011301 (2010), arXiv:1003.4505 [hep-ph].
- [6] F. Iocco, M. Taoso, F. Leclercq, and G. Meynet, *Physical Review Letters* **108**, 061301 (2012), arXiv:1201.5387 [astro-ph.SR].

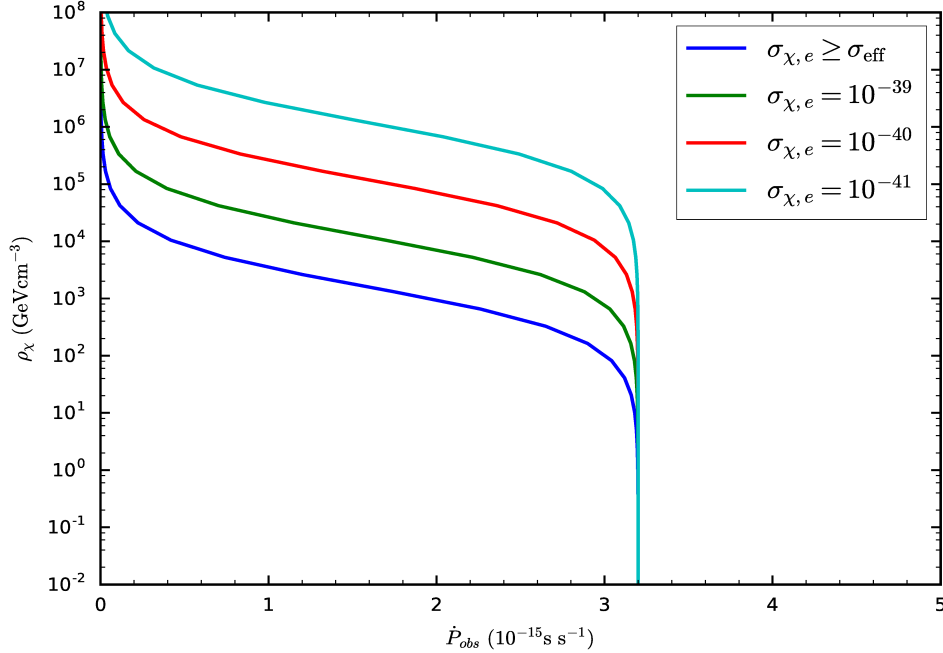


FIG. 2. The relation between \dot{P}_{obs} and WIMPs local density ρ_χ , with different $\sigma_{\chi,e}$. Note that if $\sigma_{\chi,e} \geq \sigma_{eff}$, the $\sigma_{\chi,e}$ do not influence the profile which takes the same curve as $\sigma_{\chi,e} = \sigma_{eff}$.

- [7] A. C. Vincent, P. Scott, and A. Serenelli, *Physical Review Letters* **114**, 081302 (2015), arXiv:1411.6626 [hep-ph].
- [8] I. V. Moskalenko and L. L. Wai, *Astrophys. J. Lett.* **659**, L29 (2007), astro-ph/0702654.
- [9] G. Bertone and M. Fairbairn, *Phys. Rev. D* **77**, 043515 (2008), arXiv:0709.1485.
- [10] T. J. Hurst, A. R. Zentner, A. Natarajan, and C. Badenes, *Phys. Rev. D* **91**, 103514 (2015), arXiv:1410.3925.
- [11] P. Amaro-Seoane, J. Casanellas, R. Schödel, E. Davidson, and J. Cuadra, *Mon. Not. Roy. Astron. Soc.* **459**, 695 (2016), arXiv:1512.00456.
- [12] D. E. Winget and S. O. Kepler, *Annu. Rev. Astron. Astrophys.* **46**, 157 (2008), arXiv:0806.2573.
- [13] G. Fontaine and P. Brassard, *Publications of the Astronomical Society of the Pacific* **120**, 1043 (2008).
- [14] A. Gould, *Astrophys. J.* **321**, 560 (1987).
- [15] A. Gould, *Astrophys. J.* **356**, 302 (1990).
- [16] K. Griest and D. Seckel, *Nuclear Physics B* **283**, 681 (1987).
- [17] A. Gould, *Astrophys. J.* **321**, 571 (1987).
- [18] P. Scott, M. Fairbairn, and J. Edsjö, *Mon. Not. Roy. Astron. Soc.* **394**, 82 (2009), arXiv:0809.1871.
- [19] M. Taoso, F. Iocco, G. Meynet, G. Bertone, and P. Eggenberger, *Phys. Rev. D* **82**, 083509 (2010), arXiv:1005.5711.
- [20] L. Mestel, *Mon. Not. Roy. Astron. Soc.* **112**, 583 (1952).
- [21] S. D. Kawaler, D. E. Winget, I. Iben, Jr., and C. J. Hansen, *Astrophys. J.* **302**, 530 (1986).
- [22] S. O. Kepler, O. Giovannini, M. A. Wood, and et al., *Astrophys. J.* **447**, 874 (1995).
- [23] A. S. Mukadam, A. Bischoff-Kim, O. Fraser, and et al., *Astrophys. J.* **771**, 17 (2013).
- [24] A. D. Romero, A. H. Córscico, L. G. Althaus, S. O. Kepler, B. G. Castanheira, and M. M. Miller Bertolami, *Mon. Not. Roy. Astron. Soc.* **420**, 1462 (2012), arXiv:1109.6682 [astro-ph.SR].
- [25] A. H. Córscico, L. G. Althaus, M. M. Miller Bertolami, A. D. Romero, E. García-Berro, J. Isern, and S. O. Kepler, *Mon. Not. Roy. Astron. Soc.* **424**, 2792 (2012), arXiv:1205.6180 [astro-ph.SR].
- [26] D. Merritt, G. Meylan, and M. Mayor, *Astron. J.* **114**, 1074 (1997), astro-ph/9612184.
- [27] G. van de Ven, R. C. E. van den Bosch, E. K. Verolme, and P. T. de Zeeuw, *Astron. Astrophys.* **445**, 513 (2006), astro-ph/0509228.
- [28] A. H. Córscico, L. G. Althaus, A. D. Romero, A. S. Mukadam, E. García-Berro, J. Isern, S. O. Kepler, and M. A. Corti, *J. Cosmol. Astropart. Phys.* **12**, 010 (2012), arXiv:1211.3389 [astro-ph.SR].
- [29] A. H. Córscico, A. D. Romero, L. G. Althaus, E. García-Berro, J. Isern, S. O. Kepler, M. M. Miller Bertolami, D. J. Sullivan, and P. Chote, *J. Cosmol. Astropart. Phys.* **7**, 036 (2016), arXiv:1605.06458 [astro-ph.SR].
- [30] T. Battich, A. H. Córscico, L. G. Althaus, and M. M. Miller Bertolami, *J. Cosmol. Astropart. Phys.* **8**, 062 (2016), arXiv:1605.07668 [astro-ph.SR].